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FUZZY APPROXIMATION BASED ON $\tau - \mathfrak{K}$ FUZZY OPEN (CLOSED) SETS

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Abstract. Rough set theory can be generalized by induced topology through equivalence relations. Motivated by the work of generalization of the rough set via topology, the concept and properties of $\tau - \mathfrak{K}$-fuzzy open (closed) sets are proposed. Considering the $\tau - \mathfrak{K}$-fuzzy open (closed) sets, we have obtained the $\tau - \mathfrak{K}$-fuzzy lower and upper approximations and also proved their properties. $\tau - \mathfrak{K}$-fuzzy open sets can be represented as $\tau - \mathfrak{K}$-open sets by α level sets. The properties of $\tau - \mathfrak{K}$-fuzzy approximations and fuzzy rough approximations on the basis of binary fuzzy relation are compared. Finally, an example and the decision method's algorithm to illustrate the $\tau - \mathfrak{K}$-fuzzy approximation-based approach to decision making are presented.

1. **Introduction.** Z. Pawlak [11] introduced the lower and upper approximations for uncertainty by considering equivalence classes on approximation space. K. Qin et al., Y.Y. Yao, Y.Y. Yao, T.Y. Lin, H. Zhang et al., W. Zhu, etc. [15, 26, 27, 28, 32, 33] have presented generalization of this theory. L.A. Zadeh [30] defined the fuzzy set theory to deal with uncertainty in 1965. In fuzzy set theory, elements are characterized by membership values ranging from 0 to 1, whereas rough set theory has only two outcomes 0 and 1. In 1990, D. Dubois and H. Prade [4] proposed fuzzy rough set and rough fuzzy set. A rough fuzzy set is a fuzzy set that is approximated in crisp environment. A fuzzy set's approximation in fuzzy approximation space is represented by the fuzzy rough set. Both approximations were generalized by several authors [22, 23, 25]. Topology is a branch of mathematics that studies the geometrical properties of an object that are retained under continuous deformation. The topology generated by binary relations (BRs) was first described by R. E. Smithson [17] in 1969. Many authors have studied rough topology and its generalizations [6, 7, 12, 29]. Rough topology has wide application in the area of medical events [1, 16, 19]. In 1968, C. L. Chang [3] generalized topology as fuzzy topology. It was further extended by R. Lowen [8] in 1976. In 2018, S. Mishra and R. Srivastava [9] proposed the fuzzy topology generated by binary fuzzy relations (BFRs) by considering left (right) relative fuzzy sets as a sub base. Recently, Anastassiou George A. [2] has extended theorems on fuzzy fractional approximation and real neural network operators. M. Abo-Elhamayel and Y. Yang [1] suggested a

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rough set generalization using topology in the year 2020. They developed the idea of topological rough sets using $\tau - \mathfrak{K}$-open set. We have proposed the $\tau - \mathfrak{K}$-fuzzy open (closed) sets and fuzzy approximations based on $\tau - \mathfrak{K}$ fuzzy open (closed) sets.

The format of the paper is: Some important definitions are included in Section 2. In Section 3, $\tau - \mathfrak{K}$-fuzzy open (closed) set, a new concept of $\tau - \mathfrak{K}$-fuzzy approximations, and their topological properties are proposed, the concept of α- cut is applied on $\tau - \mathfrak{K}$-fuzzy approximations and attributes of $\tau - \mathfrak{K}$-fuzzy approximations are compared to fuzzy rough approximations. A decision-making problem based on $\tau - \mathfrak{K}$-fuzzy open (closed) sets is discussed and an algorithm is proposed in Section 4. The final paragraph is the summary.

2. **Preliminaries.** Basic definitions and the concepts of fuzzy topology, fuzzy rough approximations, topology generated by binary fuzzy relations and other definitions etc. are given in this section. More definitions and properties were discussed in [13, 14, 21, 23, 24, 30].

We will denote the universal set by $\hat{U}$, fuzzy subsets of $\hat{U}$ by $\mathcal{Z}_1$ and $\mathcal{Z}_2$, and the set of collection of all fuzzy subsets of $\mathcal{Z}_1$ by $\mathcal{P}(\mathcal{Z}_1)$.

Definition 2.1. [30] Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be any two fuzzy subsets of universal set $\hat{U}$, then,

1. Subset: $\mathcal{Z}_1 \subseteq \mathcal{Z}_2 \implies \mu_{\mathcal{Z}_1}(\mathfrak{z}_1) \leq \mu_{\mathcal{Z}_2}(\mathfrak{z}_1), \forall \mathfrak{z}_1 \in \hat{U}$.
2. Union: $\mathcal{Z}_1 \cup \mathcal{Z}_2 = max\{\mu_{\mathcal{Z}_1}(\mathfrak{z}_1), \mu_{\mathcal{Z}_2}(\mathfrak{z}_1)\}, \forall \mathfrak{z}_1 \in \hat{U}$.
3. Intersection: $\mathcal{Z}_1 \cap \mathcal{Z}_2 = min\{\mu_{\mathcal{Z}_1}(\mathfrak{z}_1), \mu_{\mathcal{Z}_2}(\mathfrak{z}_1)\}, \forall \mathfrak{z}_1 \in \hat{U}$.
4. Complement: $\mathcal{Z}_1^c = 1 - \mu_{\mathcal{Z}_1}(\mathfrak{z}_1), \forall \mathfrak{z}_1 \in \hat{U}$.

Definition 2.2. [5, 31] A fuzzy set $\mathfrak{K}$ is called BFR if it maps each element from $\mathcal{Z}_1 \times \mathcal{Z}_2$ to $[0, 1]$, *i.e.* $\mathfrak{K} = \{(\mathfrak{z}_1, \mathfrak{z}_2), \mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_2))| \ (\mathfrak{z}_1, \mathfrak{z}_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2\}$ is a BFR in $\mathcal{Z}_1 \times \mathcal{Z}_2$.

α-level set ($\mathfrak{K}_\alpha$) and strong α-level set ($\mathfrak{K}_{\alpha^+}$) of $\mathfrak{K}$ are defined as:
$\mathfrak{K}_\alpha = \{(\mathfrak{z}_1, \mathfrak{z}_2) : \mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_2) \geq \alpha\}$,
$\mathfrak{K}_\alpha^+ = \{(\mathfrak{z}_1, \mathfrak{z}_2) : \mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_2) > \alpha\}$, where $\alpha \in [0, 1]$.
$\mathfrak{K}$ is reflexive if $\mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_1) = 1$ and ϵ-reflexive if $\mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_1) \geq \epsilon$, where $0 < \epsilon < 1$.
$\mathfrak{K}$ is symmetric if $\mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_2) = \mu_{\mathfrak{K}}(\mathfrak{z}_2, \mathfrak{z}_1)$.
$\mathfrak{K}$ is transitive if $\mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_3) \geq \underset{\mathfrak{z}_2}{max}\ min[\mu_{\mathfrak{K}}(\mathfrak{z}_1, \mathfrak{z}_2), \mu_{\mathfrak{K}}(\mathfrak{z}_2, \mathfrak{z}_3)], \forall \mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3 \in \mathfrak{K}$,
i.e.$\mathfrak{K} \circ \mathfrak{K} \subset \mathfrak{K}$, where $\mathfrak{K} \circ \mathfrak{K} = \mathfrak{K}^2$ is called max-min composition.

Definition 2.3. [3] A collection τ of fuzzy sets in the universal set $\hat{U}$ is called fuzzy topology if it satisfies the following criteria:

1. The empty fuzzy set with membership value 0 and the universal fuzzy set with membership value 1 belongs to τ.
2. Finite intersection of fuzzy sets belonging to τ also belongs to τ.
3. Arbitrary union of fuzzy sets belonging to τ also belongs to τ.

   $(\hat{U}, \tau)$ is called the fuzzy topological space (FTS).

Remark 1. All the components of τ are described as fuzzy open sets and fuzzy closed sets are their complement.

Definition 2.4. [3] Let $\mathcal{Z}_1$ be any fuzzy subset of universal set $\hat{U}$ and $(\hat{U}, \tau)$ be any FTS, then $\mathcal{Z}_1$ is said to be the fuzzy neighborhood (nbd) of a fuzzy set $\mathcal{A}$, if there exists a fuzzy open set $\mathcal{G}_1$ such that $\mathcal{A} \subseteq \mathcal{G}_1 \subseteq \mathcal{Z}_1$.

Remark 2. In crisp we consider nbd of points where as in fuzzy we consider nbd of fuzzy sets.

Definition 2.5. [24] Fuzzy interior (FI) and fuzzy closure (FC): Let $(\hat{U}, \tau)$ be a FTS and $\mathcal{Z}_1$ be any fuzzy subset of $\hat{U}$, then FI $(int(\mathcal{Z}_1))$ and FC $(cl(\mathcal{Z}_1))$ of $\mathcal{Z}_1$ are defined as: $int(\mathcal{Z}_1) = \cup\{\mathcal{G}_1 : \mathcal{G}_1$ is an element of τ and $\mathcal{G}_1 \subseteq \mathcal{Z}_1\}$, $cl(\mathcal{Z}_1) = \cap\{\mathcal{G}_1^c : \mathcal{G}_1^c$ is fuzzy closed set and $\mathcal{Z}_1 \subseteq \mathcal{G}_1^c\}$.

Definition 2.6. [4, 13] For any BFR $\mathfrak{K}$ and any fuzzy subset $\mathcal{Z}_1$ of $\hat{U}$, fuzzy approximation space is represented by $(\hat{U}, \mathfrak{K})$. Then, the fuzzy lower and upper approximations of $\mathcal{Z}_1$ are defined as: $\underline{\mathfrak{K}}(\mathcal{Z}_1)(\mathfrak{z}_1) = \wedge_{\mathfrak{z}_2 \in \hat{U}}(\mathcal{Z}_1(\mathfrak{z}_2) \vee (1 - \mathfrak{K}(\mathfrak{z}_1, \mathfrak{z}_2)))$, $\overline{\mathfrak{K}}(\mathcal{Z}_1)(\mathfrak{z}_1) = \vee_{\mathfrak{z}_2 \in \hat{U}}(\mathcal{Z}_1(\mathfrak{z}_2) \wedge \mathfrak{K}(\mathfrak{z}_1, \mathfrak{z}_2))$, respectively, where $\mathfrak{z}_1, \mathfrak{z}_2 \in \hat{U}$. $(\underline{\mathfrak{K}}(\mathcal{Z}_1), \overline{\mathfrak{K}}(\mathcal{Z}_1))$ is the fuzzy rough set of $\mathcal{Z}_1$.

Definition 2.7. [9] Let $\mathfrak{K}$ be any BFR on the universal set $\hat{U}$, then $\mathfrak{r}_{\mathfrak{K}}(\mathfrak{z}_1) = \{(\mathfrak{z}_2, \mu_{\mathfrak{z}_2})|\ \mathfrak{z}_2 \in \hat{U}, \mathfrak{z}_1 \mathfrak{K} \mathfrak{z}_2\}$ is called the right $\mathfrak{K}$-relative fuzzy set, and $\mathfrak{l}_{\mathfrak{K}}(\mathfrak{z}_1) = \{(\mathfrak{z}_2, \mu_{\mathfrak{z}_2})|\ \mathfrak{z}_2 \in \hat{U}, \mathfrak{z}_2 \mathfrak{K} \mathfrak{z}_1\}$ is called the left $\mathfrak{K}$-relative fuzzy set.

Theorem 2.8. [20] *Let $(\hat{U}, \tau)$ be FTS and $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be fuzzy subsets of $\hat{U}$, then the following results for fuzzy interior and fuzzy closure hold:*

1. $cl(\mathcal{Z}_1 \cup \mathcal{Z}_2) = cl(\mathcal{Z}_1) \cup cl(\mathcal{Z}_2)$,
2. $cl(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq cl(\mathcal{Z}_1) \cap cl(\mathcal{Z}_2)$,
3. $int(\mathcal{Z}_1 \cap \mathcal{Z}_2) = int(\mathcal{Z}_1) \cap int(\mathcal{Z}_2)$,
4. $int(\mathcal{Z}_1) \cup int(\mathcal{Z}_2) \subseteq int(\mathcal{Z}_1 \cup \mathcal{Z}_2)$,
5. *If* $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, *then* $cl(\mathcal{Z}_1) \subseteq cl(\mathcal{Z}_2)$ *and* $int(\mathcal{Z}_1) \subseteq int(\mathcal{Z}_2)$,
6. $cl(\mathcal{Z}_1^c) = (int(\mathcal{Z}_1))^c$,
7. $int(\mathcal{Z}_1)^c = (cl(\mathcal{Z}_1))^c$.

3. Proposed work.

Here, we have introduced $\tau - \mathfrak{K}$-fuzzy open(closed) set, which is based on fuzzy topology and define the $\tau - \mathfrak{K}$-fuzzy approximation based on $\tau - \mathfrak{K}$-fuzzy open(closed) sets. These approximations are defined from a collection of $\tau - \mathfrak{K}$-fuzzy open(closed) sets.

3.1. $\tau - \mathfrak{K}$-fuzzy open set.

Definition 3.1. **$\tau - \mathfrak{K}$-fuzzy open set:** For any FTS $(\hat{U}, \tau)$, let $\mathfrak{K}$ be any BFR on universal set $\hat{U}$. A fuzzy subset $\mathcal{G}_{\mathfrak{K}}$ of $\hat{U}$ is called $\tau - \mathfrak{K}$-fuzzy open set (FOS) if there exists non empty FOS $\mathcal{G}_1$ such that $\mathcal{G}_1 \subseteq cl(\mathcal{G}_{\mathfrak{K}})$. $\tau - \mathfrak{K}$-fuzzy closed set (FCS) is the complement of $\tau - \mathfrak{K}$-FOS.

Example 3.2. Let $\hat{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\}$ be universal set and $\mathcal{Z}_1 = \{(\mathfrak{u}_1, .5), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, .4)\}$ be a fuzzy subset of $\hat{U}$ and $\tau = \Big\{\{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, 0), (\mathfrak{u}_3, 0)\}, \{(\mathfrak{u}_1, 1), (\mathfrak{u}_2, 1), (\mathfrak{u}_3, 1)\}, \{(\mathfrak{u}_1, .5), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, 0)\}, \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, 0), (\mathfrak{u}_3, .4)\}, \{(\mathfrak{u}_1, .5), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, .4)\}\Big\}$ be fuzzy topology on $\hat{U}$ w.r.t. $\mathcal{Z}_1$ and $\tau^c = \Big\{\{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, 0), (\mathfrak{u}_3, 0)\}, \{(\mathfrak{u}_1, 1), (\mathfrak{u}_2, 1), (\mathfrak{u}_3, 1)\}, \{(\mathfrak{u}_1, .5), (\mathfrak{u}_2, .3), (\mathfrak{u}_3, 1)\}, \{(\mathfrak{u}_1, 1), (\mathfrak{u}_2, 1), (\mathfrak{u}_3, .6)\}, \{(\mathfrak{u}_1, .5), (\mathfrak{u}_2, .3), (\mathfrak{u}_3, .6)\}\Big\}$.

Then, $\mathcal{Z}_2 = \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, 0)\}$ is a $\tau - \mathfrak{K}$-FOS as there exists non empty FOS $\mathcal{G}_1 = \{(\mathfrak{u}_1, .5), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, 0)\}$, such that $\mathcal{G}_1 \subseteq cl(\mathcal{Z}_2)$, where $cl(\mathcal{Z}_2) = \{(\mathfrak{u}_1, 1), (\mathfrak{u}_2, 1), (\mathfrak{u}_3, .6)\}$.

Hence, $\mathcal{Z}_2 = \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, 0)\}$ is $\tau - \mathfrak{K}$-FOS. (using definition 3.1)

Theorem 3.3. *In a FTS, every FOS is $\tau-\mathfrak{K}$-FOS, but every FOS may not be $\tau-\mathfrak{K}$-FOS.*

Proof. Take $(\hat{U},\tau)$ a FTS and $\mathcal{G}_1$ be a FOS. Every set is contained in its closure, hence $\mathcal{G}_1 \subseteq cl(\mathcal{G}_1)$. Therefore, $\mathcal{G}_1$ is $\tau-\mathfrak{K}$-FOS. (using definition 3.1)

Converse: In example 3.2, we can see that $\mathcal{Z}_2 = \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, 0)\}$ is $\tau-\mathfrak{K}$-FOS but not FOS, as it is not the element of τ. □

Theorem 3.4. *A fuzzy subset $\mathcal{Z}_1$ of $\hat{U}$ is called $\tau-\mathfrak{K}$-FCS in FTS $(\hat{U},\tau)$ iff $\exists$ a FCS $\mathcal{F} \subset \hat{U} \ni int(\mathcal{Z}_1) \subseteq \mathcal{F}$.*

Proof. Let $\mathcal{Z}_1$ be a $\tau-\mathfrak{K}$-FCS, then $\mathcal{Z}_1^c$ is $\tau-\mathfrak{K}$-FOS. This implies that $\exists$ a non empty FOS $\mathcal{G}_1 \ni \mathcal{G}_1 \subseteq cl(\mathcal{Z}_1^c)$.

Therefore, $(cl(\mathcal{Z}_1^c))^c \subseteq \mathcal{G}_1^c$. Since $cl(\mathcal{Z}_1^c) = (int(\mathcal{Z}_1))^c$. (using theorem 2.8)
$\implies int(\mathcal{Z}_1) \subseteq \mathcal{G}_1^c$.

Let $\mathcal{F} = \mathcal{G}_1^c$, where $\mathcal{G}_1^c$ is fuzzy closed set.

Hence, $int(\mathcal{Z}_1) \subseteq \mathcal{F}$.

Converse: let $\mathcal{Z}_1$ be fuzzy subset of $\hat{U}$ and $\exists$ FCS $\mathcal{F} \subset \hat{U} \ni int(\mathcal{Z}_1) \subseteq \mathcal{F}$.
$\implies \mathcal{F}^c \subseteq (int(\mathcal{Z}_1))^c$.
$\implies \mathcal{F}^c \subseteq cl(\mathcal{Z}_1^c)$. Since $cl(\mathcal{Z}_1^c) = (int(\mathcal{Z}_1))^c$. (using theorem 2.8)

Since, $\mathcal{F}^c$ is FOS and $\mathcal{F}^c \subseteq cl(\mathcal{Z}_1^c) \implies \mathcal{Z}_1^c$ is $\tau-\mathfrak{K}$ -FOS.
$\implies \mathcal{Z}_1$ is $\tau-\mathfrak{K}$ -FCS. □

Proposition 3.5. *Every fuzzy neighborhood of any fuzzy set in FTS $(\hat{U},\tau)$ is $\tau-\mathfrak{K}$-FOS.*

Proof. Let $\mathcal{N}_{\mathcal{Z}_1}$ be the fuzzy neighborhood of a non-empty fuzzy subset $\mathcal{Z}_1$ of $\hat{U}$. Then, there exist at least one FOS $\mathcal{G}_1$ such that $\mathcal{Z}_1 \subseteq \mathcal{G}_1 \subseteq \mathcal{N}_{\mathcal{Z}_1} \subseteq cl(\mathcal{N}_{\mathcal{Z}_1})$. Hence, by using definition 3.1, $\mathcal{N}_{\mathcal{Z}_1}$ is $\tau-\mathfrak{K}$-FOS. □

Proposition 3.6. *If $\mathcal{Z}_1$ is a $\tau-\mathfrak{K}$-FOS in FTS $(\hat{U},\tau)$ and $\mathcal{Z}_2$ is any fuzzy subset of $\hat{U}$, then $\mathcal{Z}_1 \cup \mathcal{Z}_2$ is also $\tau-\mathfrak{K}$-FOS.*

Proof. Consider $\mathcal{Z}_1$ be a $\tau-\mathfrak{K}$-FOS and $\mathcal{Z}_2$ be any fuzzy subset of $\hat{U}$, then there exists non empty FOS $\mathcal{G}_1 \ni \mathcal{G}_1 \subseteq cl(\mathcal{Z}_1)$. (using definition 3.1)
$\implies \mathcal{G}_1 \subseteq cl(\mathcal{Z}_1) \cup cl(\mathcal{Z}_2)$.

We know that, $cl(\mathcal{Z}_1 \cup \mathcal{Z}_2) \subseteq cl(\mathcal{Z}_1) \cup cl(\mathcal{Z}_2)$, (using theorem 2.8).

Therefore, $\mathcal{G}_1 \subseteq cl(\mathcal{Z}_1 \cup \mathcal{Z}_2)$. Hence, $\mathcal{Z}_1 \cup \mathcal{Z}_2$ is $\tau-\mathfrak{K}$-FOS. □

Theorem 3.7. *Arbitrary union of $\tau-\mathfrak{K}$-FOSs is also $\tau-\mathfrak{K}$-FOS.*

Proof. Let $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, ...$ be $\tau-\mathfrak{K}$-FOSs, then there exists non empty FOSs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, ... \ni \mathcal{G}_1 \subseteq cl(\mathcal{Z}_1), \mathcal{G}_2 \subseteq cl(\mathcal{Z}_2), \mathcal{G}_3 \subseteq cl(\mathcal{Z}_3), ...$ (using definition 3.1)
$\implies (\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup ...) \subseteq (cl(\mathcal{Z}_1) \cup (cl(\mathcal{Z}_2) \cup (cl(\mathcal{Z}_3)...)$.
$\implies (\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup ...) \subseteq cl(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup ...)$. (using theorem 2.8)

Since, arbitrary union of FOSs is open and contained in the closure of arbitrary union of $\tau-\mathfrak{K}$- FOS. Therefore, $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup ...$ is also $\tau-\mathfrak{K}$-FOS.

Remark 3. $(\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_3 \cap .... \cap \mathcal{Z}_n)$ may or may not be $\tau-\mathfrak{K}$-FOS whenever $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3.......\mathcal{Z}_n$ are $\tau-\mathfrak{K}$-FOSs. □

Example 3.8. In example 3.2, take two fuzzy subsets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} = \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, 0)\}$ and $\mathcal{B} = \{(\mathfrak{u}_1, .4), (\mathfrak{u}_2, 0), (\mathfrak{u}_3, 0)\}$.

We get, $cl(\mathcal{A}) = \{(\mathfrak{u}_1, 1), (\mathfrak{u}_2, 1), (\mathfrak{u}_3, .6)\}$ and $cl(\mathcal{B}) = \{(\mathfrak{u}_1, 0.5), (\mathfrak{u}_2, .3), (\mathfrak{u}_3, .6)\}$. There exists non empty FOS,

$\mathcal{G}_1 = \{(\mathfrak{u}_1, .5), (\mathfrak{u}_2, .7), (\mathfrak{u}_3, 0)\} \subseteq cl(\mathcal{A})$ and $\mathcal{G}_2 = \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, 0), (\mathfrak{u}_3, .4)\} \subseteq cl(\mathcal{B})$.

Therefore, $\mathcal{A}$ and $\mathcal{B}$ both are $\tau - \mathfrak{K}$-FOSs. (using definition 3.1)

$\mathcal{A} \cap \mathcal{B} = \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, 0), (\mathfrak{u}_3, 0)\}$ and $cl(\mathcal{A} \cap \mathcal{B}) = \{(\mathfrak{u}_1, 0), (\mathfrak{u}_2, 0), (\mathfrak{u}_3, 0)\}$.

Therefore, there are no non- empty FOS contained in $cl(\mathcal{A} \cap \mathcal{B})$.

Hence $\mathcal{A} \cap \mathcal{B}$ is not $\tau - \mathfrak{K}$-FOS.

Remark 4. From example 3.8 and remark 3, it is clear that $\tau - \mathfrak{K}$-FOSs do not satisfy all the axioms of fuzzy topology.

Theorem 3.9. *Arbitrary Intersection of $\tau - \mathfrak{K}$-FCSs is also $\tau - \mathfrak{K}$-FCS.*

Proof. Consider an arbitrary family of $\tau - \mathfrak{K}$-FCSs as $\{\mathcal{F}_i : i \in I\}$. Then, $\{\mathcal{F}_i^c : i \in I\}$ is called the family of $\tau - \mathfrak{K}$-FOSs.

Since, arbitrary union of $\tau - \mathfrak{K}$-FOSs is also $\tau - \mathfrak{K}$-FOS. Hence, $\cup_{i \in I} \mathcal{F}_i^c$ is $\tau - \mathfrak{K}$-FOS. Therefore, $\cap_{i \in I} \mathcal{F}_i$ is $\tau - \mathfrak{K}$-FCS (by De Morgan's Law). □

For $\tau - \mathfrak{K}$-FOS (FCS), we can extend definition 2.5 as:

Definition 3.10. Let $(\hat{U}, \tau)$ be FTS and $\mathcal{Z}_1$ be any fuzzy subset of $\hat{U}$, then $\tau - \mathfrak{K}$-FI and $\tau - \mathfrak{K}$-FC of $\mathcal{Z}_1$ are defined as: $int_{\mathfrak{K}}(\mathcal{Z}_1) = \cup\{\mathcal{G}_{\mathfrak{K}} : \mathcal{G}_{\mathfrak{K}}$ is $\tau - \mathfrak{K}$-FOS, $\mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_1\}, cl_{\mathfrak{K}}(\mathcal{Z}_1) = \cap\{\mathcal{G}_{\mathfrak{K}}^c : \mathcal{G}_{\mathfrak{K}}^c$ is $\tau - \mathfrak{K}$-FCS, $\mathcal{Z}_1 \subseteq \mathcal{G}_{\mathfrak{K}}^c\}$ respectively.

Properties of $\tau - \mathfrak{K}$-FI and $\tau - \mathfrak{K}$-FC are proved in theorem 3.9.

Theorem 3.11. *For any FTS $(\hat{U}, \tau)$ and for any fuzzy subset $\mathcal{Z}_1$ of $\hat{U}$, the following results hold:*

1. $\mathcal{Z}_1 \subseteq cl_{\mathfrak{K}}(\mathcal{Z}_1)$,
2. $\mathcal{Z}_1 = cl_{\mathfrak{K}}(\mathcal{Z}_1)$ *iff* $\mathcal{Z}_1$ *is* $\tau - \mathfrak{K}$-*FCS,*
3. $int_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \mathcal{Z}_1$,
4. $int_{\mathfrak{K}}(\mathcal{Z}_1) = \mathcal{Z}_1$ *iff* $\mathcal{Z}_1$ *is* $\tau - \mathfrak{K}$-*FOS,*
5. $(int_{\mathfrak{K}}(\mathcal{Z}_1))^c = cl_{\mathfrak{K}}(\mathcal{Z}_1^c)$,
6. $(cl_{\mathfrak{K}}(\mathcal{Z}_1))^c = int_{\mathfrak{K}}(\mathcal{Z}_1^c)$.

Proof. Results 1,2,3,4 can be proved using definition 3.10.

For result 5, $int_{\mathfrak{K}}(\mathcal{Z}_1) = \cup\{\mathcal{G}_{\mathfrak{K}} : \mathcal{G}_{\mathfrak{K}}$ is $\tau - \mathfrak{K}$-FOS, $\mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_1\}$, taking complement on both sides we get, $(int_{\mathfrak{K}}(\mathcal{Z}_1))^c = \cap\{\mathcal{G}_{\mathfrak{K}}^c : \mathcal{G}_{\mathfrak{K}}^c$ is $\tau - \mathfrak{K}$-FCS, $\mathcal{Z}_1^c \subseteq \mathcal{G}_{\mathfrak{K}}^c\} = cl_{\mathfrak{K}}(\mathcal{Z}_1^c)$.

Similarly, we can prove result 6. □

3.2. **$\tau - \mathfrak{K}$-fuzzy approximation based on $\tau - \mathfrak{K}$-fuzzy open(closed) sets.** Characteristics of $\tau - \mathfrak{K}$- fuzzy lower $(FL_{\tau-\mathfrak{K}})$ and $\tau - \mathfrak{K}$- fuzzy upper $(FU_{\tau-\mathfrak{K}})$ approximations are examined in this part and presented a notion of these approximations based on generated fuzzy topology.

Definition 3.12. Let $\mathfrak{K}$ be any BFR on the universal set $\hat{U}$ and τ be the fuzzy topology generated by $\mathfrak{K}$, then $(\hat{U}, \tau, \mathfrak{K})$ is called the fuzzy topological approximation space (FTAS).

Let $\mathcal{Z}_1$ be any fuzzy subset of universal set $\hat{U}$. Then, $(FL_{\tau-\mathfrak{K}})$ and $(FU_{\tau-\mathfrak{K}})$ approximations of $\mathcal{Z}_1$ can be defined by $\tau - \mathfrak{K}$-FI and $\tau - \mathfrak{K}$-FC as:

i $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = int_{\mathfrak{K}}(\mathcal{Z}_1) = \cup\{\mathcal{G}_{\mathfrak{K}} : \mathcal{G}_{\mathfrak{K}}$ is $\tau - \mathfrak{K}$-FOS, $\mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_1\}$,

ii $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = cl_{\mathfrak{K}}(\mathcal{Z}_1) = \cap\{\mathcal{G}_{\mathfrak{K}}^c : \mathcal{G}_{\mathfrak{K}}^c$ is $\tau - \mathfrak{K}$-FCS, $\mathcal{Z}_1 \subseteq \mathcal{G}_{\mathfrak{K}}^c\}$ respectively.

Example 3.13. Let $\hat{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\}$ be an universal set, $\mathcal{Z}_1 = \left\{\frac{.4}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}$ be a fuzzy subset of $\hat{U}$ and $\mathfrak{R} = \begin{bmatrix} .5 & .4 & .7 \\ .4 & 0 & 0 \\ 1 & 1 & .7 \end{bmatrix}$ be BFR on $\hat{U}$.

$\mathfrak{l}_{\mathfrak{R}}(\mathfrak{u}_1) = \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}$, $\mathfrak{l}_{\mathfrak{R}}(\mathfrak{u}_2) = \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}$, $\mathfrak{l}_{\mathfrak{R}}(\mathfrak{u}_3) = \left\{\frac{.7}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}$ are left $\mathfrak{R}$- relative fuzzy sets.

$\mathfrak{r}_{\mathfrak{R}}(\mathfrak{u}_1) = \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}$, $\mathfrak{r}_{\mathfrak{R}}(\mathfrak{u}_2) = \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}$, $\mathfrak{r}_{\mathfrak{R}}(\mathfrak{u}_3) = \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}$ are right $\mathfrak{R}$- relative fuzzy sets.

Let the collection of left $\mathfrak{R}$- relative fuzzy sets be denoted by $\mathcal{S}_1$, collection of right $\mathfrak{R}$- relative fuzzy sets be denoted by $\mathcal{S}_2$ and collection of all left and right $\mathfrak{R}$-relative fuzzy sets be denoted by $\mathcal{S}$.

Fuzzy topology generated by $\mathcal{S}_1$ is given as: $\tau_1 = \Big\{\left\{\frac{0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.7}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.7}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.7}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\} + \left\{\frac{.5}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}\Big\}$ and $\tau_1^c = \Big\{\left\{\frac{0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.6}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.3}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.6}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.3}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.3}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}\Big\}$.

Fuzzy topology generated by $\mathcal{S}_2$ is given as: $\tau_2 = \Big\{\left\{\frac{0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}\Big\}$ and $\tau_2^c = \Big\{\left\{\frac{0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.6}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}\Big\}$.

Fuzzy topology generated by $\mathcal{S}$ is given as: $\tau = \Big\{\left\{\frac{0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.7}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.4}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}, \left\{\frac{.7}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.7}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.7}{\mathfrak{u}_1} + \frac{.4}{\mathfrak{u}_2} + \frac{.7}{\mathfrak{u}_3}\right\}\Big\}$ and $\tau^c = \Big\{\left\{\frac{0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{1}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.6}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.3}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.6}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{1}{\mathfrak{u}_3}\right\}, \left\{\frac{.0}{\mathfrak{u}_1} + \frac{0}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.6}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}, \left\{\frac{.3}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.3}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.5}{\mathfrak{u}_1} + \frac{1}{\mathfrak{u}_2} + \frac{0}{\mathfrak{u}_3}\right\}, \left\{\frac{.3}{\mathfrak{u}_1} + \frac{.6}{\mathfrak{u}_2} + \frac{.3}{\mathfrak{u}_3}\right\}\Big\}$.

In Table-1, we determine the $\tau_1 - \mathfrak{R}$-FOSs, $\tau_1 - \mathfrak{R}$-FCSs, $FL_{\tau-\mathfrak{R}}$ and $FU_{\tau-\mathfrak{R}}$ approximations based on $\tau_1 - \mathfrak{R}$-FOSs (FCS).

In Table-2, we determine the $\tau_2 - \mathfrak{R}$-FOSs, $\tau_2 - \mathfrak{R}$-FCSs, $FL_{\tau-\mathfrak{R}}$ and $FU_{\tau-\mathfrak{R}}$ approximations based on $\tau_2 - \mathfrak{R}$-FOSs(FCSs).

In Table-3, we determine the $\tau - \mathfrak{R}$-FOSs, $\tau - \mathfrak{R}$-FCSs, $FL_{\tau-\mathfrak{R}}$ and $FU_{\tau-\mathfrak{R}}$ approximations based on $\tau - \mathfrak{R}$-FOSs(FCSs).

TABLE 1.

$\mathcal{A} = \mathcal{P}(\mathcal{Z}_1)$	$cl(\mathcal{A})$	$\tau_1 - \mathfrak{K}$-FOS or not	$\tau_1 - \mathfrak{K}$-FCS	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A})$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A})$
$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.3}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.3}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{1}{u_2}+\frac{.3}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$

TABLE 2.

$\mathcal{A} = \mathcal{P}(\mathcal{Z}_1)$	$cl(\mathcal{A})$	$\tau_2 - \mathfrak{K}$-FOS or not	$\tau_2 - \mathfrak{K}$-FCS	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A})$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A})$
$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{1}{u_1}+\frac{.4}{u_2}+\frac{1}{u_3}\right\}$	$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{1}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{1}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$	$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$

TABLE 3.

$\mathcal{A} = \mathcal{P}(\mathcal{Z}_1)$	$cl(\mathcal{A})$	$\tau - \mathfrak{K}$-FOS or not	$\tau - \mathfrak{K}$-FCS	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A})$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A})$
$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.3}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{1}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{0}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.3}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	No	-	$\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{0}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$
$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.5}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	Yes	$\left\{\frac{.6}{u_1}+\frac{.4}{u_2}+\frac{.7}{u_3}\right\}$	$\left\{\frac{.4}{u_1}+\frac{.6}{u_2}+\frac{.3}{u_3}\right\}$	$\left\{\frac{.6}{u_1}+\frac{1}{u_2}+\frac{.7}{u_3}\right\}$

Remark 5. $\left\{\frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}\right\}$ and $\left\{\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right\}$ are considered as $\tau - \mathfrak{K}$-FOSs(FCSs).

Lemma 3.14. *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be any two fuzzy subsets of universal set $\hat{U}$ and $(\hat{U}, \tau, \mathfrak{K})$ be a FTAS, then the following properties hold:*

1. $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \mathcal{Z}_1 \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$,
2. $\underline{\mathfrak{K}}_{\mathfrak{K}}(\phi) = \phi = \overline{\mathfrak{K}}_{\mathfrak{K}}(\phi)$,
3. $\underline{\mathfrak{K}}_{\mathfrak{K}}(\hat{U}) = \hat{U} = \overline{\mathfrak{K}}_{\mathfrak{K}}(\hat{U})$,
4. $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = \left(\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1^c)\right)^c$,
5. $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = \left(\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1^c)\right)^c$,
6. $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$,
7. $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2) = \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$,
8. $\mathcal{Z}_1 \subseteq \mathcal{Z}_2 \Rightarrow \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$ *and* $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$,
9. $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$,
10. $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.

Proof. **Property 1.** $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = int_{\mathfrak{K}}(\mathcal{Z}_1)$ (using definition 3.12) and $\tau - \mathfrak{K}$-FI of $\mathcal{Z}_1$ is the biggest open set contained in $\mathcal{Z}_1$ implies $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \mathcal{Z}_1$.

$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = cl_{\mathfrak{K}}(\mathcal{Z}_1)$ (using definition 3.12) and $\tau - \mathfrak{K}$-FC of $\mathcal{Z}_1$ is the smallest open set that contains the fuzzy set $\mathcal{Z}_1$ Therefore, $\mathcal{Z}_1 \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$.

Properties 2,3,4,5 can be proved using definition 3.12.

Property 6. Let $\mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2)$, then $\exists$ a $\tau-\mathfrak{K}$-FOS $\mathcal{G}_{\mathfrak{K}} \ni \mathcal{A} \subseteq \mathcal{G}_{\mathfrak{K}} \subseteq (\mathcal{Z}_1 \cap \mathcal{Z}_2)$.
$\Rightarrow \mathcal{A} \subseteq \mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_1$ and $\mathcal{A} \subseteq \mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_2$.
$\Rightarrow \mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$ and $\mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
$\Rightarrow \mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
Similarly, we can prove that $\Rightarrow \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2)$.
Therefore, $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.

Property 7. Let $\mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$, then $\exists$ a $\tau - \mathfrak{K}$-FCS $\mathcal{G}^c_{\mathfrak{K}} \ni \mathcal{A} \subseteq \mathcal{G}^c_{\mathfrak{K}}$ and $(\mathcal{Z}_1 \cup \mathcal{Z}_2) \cap \mathcal{G}^c_{\mathfrak{K}} \neq \phi$.
$\Rightarrow (\mathcal{Z}_1 \cap \mathcal{G}^c_{\mathfrak{K}}) \cup (\mathcal{Z}_2 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$.
$\Rightarrow (\mathcal{Z}_1 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$ or $(\mathcal{Z}_2 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$.
$\Rightarrow \mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$ or $\mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
$\Rightarrow \mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
$\Rightarrow \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
Similarly, we can prove that $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$.
Therefore, $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2) = \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.

Property 8. Consider $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ and $\mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$, then $\exists$ $\tau - \mathfrak{K}$-FOS $\mathcal{G}_{\mathfrak{K}} \ni \mathcal{A} \subseteq \mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_1$.
$\Rightarrow \mathcal{A} \subseteq \mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_2$.
$\Rightarrow \mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
Hence, $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
Again, $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ and $\mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$, then $\exists$ $\tau - \mathfrak{K}$-FCS $\mathcal{G}^c_1 \ni \mathcal{A} \subseteq \mathcal{G}^c_{\mathfrak{K}}$ and $(\mathcal{Z}_1 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$.
$\Rightarrow (\mathcal{Z}_2 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$.
$\Rightarrow \mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
Hence, $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.

Property 9. Let $\mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$, then either $\mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$ or $\mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$, then $\exists$ $\tau - \mathfrak{K}$-FOS $\mathcal{G}_{\mathfrak{K}} \ni \mathcal{A} \subseteq \mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_1$ or $\mathcal{A} \subseteq \mathcal{G}_{\mathfrak{K}} \subseteq \mathcal{Z}_2$.
Therefore, $\mathcal{G}_{\mathfrak{K}}$ is subset of $(\mathcal{Z}_1 \cup \mathcal{Z}_2)$.
$\Rightarrow \mathcal{A} \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$.
$\Rightarrow \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$.
Conversely, in example 3.13, it can be seen that $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ is not the subset of $\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.

Property 10. Let $\mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2)$, then $\exists$ $\tau - \mathfrak{K}$-FCS $\mathcal{G}^c_{\mathfrak{K}} \ni \mathcal{A} \subseteq \mathcal{G}^c_{\mathfrak{K}}$ and $(\mathcal{Z}_1 \cap \mathcal{Z}_2) \cap \mathcal{G}^c_{\mathfrak{K}} \neq \phi$.
$\Rightarrow (\mathcal{Z}_1 \cap \mathcal{G}^c_{\mathfrak{K}}) \cap (\mathcal{Z}_2 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$.
$\Rightarrow (\mathcal{Z}_1 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$ and $(\mathcal{Z}_2 \cap \mathcal{G}^c_{\mathfrak{K}}) \neq \phi$.
$\Rightarrow \mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$ and $\mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
$\Rightarrow \mathcal{A} \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
$\Rightarrow \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$.
Conversely, in example 3.13, it can be seen that $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$ is not the subset of $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2)$. □

3.3. **$\tau - \mathfrak{K}$-fuzzy approximations using $\alpha-$ level set.** $\tau - \mathfrak{K}$-fuzzy approximations fuzzy approximations can be converted to $\tau - \mathfrak{K}$-approximations using $\alpha-$cut on $\tau - \mathfrak{K}$-fuzzy open (closed) sets and can be represented as:

Let $(\hat{U}, \tau, \mathfrak{K})$ be FTAS and $\mathfrak{K}$ be BFR on $\hat{U}$, then $\mathfrak{K}_\alpha$ and $\mathfrak{K}_{\alpha^+}$ represent the crisp binary relations based on $\alpha-$ level and strong $\alpha-$ level sets respectively. We can generate two approximation spaces $(\hat{U}, \tau, \mathfrak{K}_\alpha)$ and $(\hat{U}, \tau, \mathfrak{K}_{\alpha^+})$ by these binary relations. The $\tau - \mathfrak{K}$-lower and $\tau - \mathfrak{K}$-upper approximations are defined as:

1. $\underline{\mathfrak{K}}_{\mathfrak{K}_\alpha}(\mathcal{Z}_1) = int_{\mathfrak{K}_\alpha}(\mathcal{Z}_1)$, $\underline{\mathfrak{K}}_{\mathfrak{K}_{\alpha^+}}(\mathcal{Z}_1) = int_{\mathfrak{K}_{\alpha^+}}(\mathcal{Z}_1)$,
2. $\overline{\mathfrak{K}}_{\mathfrak{K}_\alpha}(\mathcal{Z}_1) = cl_{\mathfrak{K}_\alpha}(\mathcal{Z}_1)$, $\overline{\mathfrak{K}}_{\mathfrak{K}_{\alpha^+}}(\mathcal{Z}_1) = cl_{\mathfrak{K}_{\alpha^+}}(\mathcal{Z}_1)$ respectively.

Remark 6. $\alpha-$ cut set converts BFRs to crisp BRs.

3.4. **Comparison of $\tau - \mathfrak{K}$-fuzzy approximation with fuzzy rough approximation.** The fuzzy rough approximations are obtained by the binary fuzzy relations established on the universal set $\hat{U}$ for any fuzzy collection. $\tau - \mathfrak{K}$-fuzzy approximations based on $\tau - \mathfrak{K}$-FOS (FCS) are defined via, $\tau - \mathfrak{K}$–FI and $\tau - \mathfrak{K}$-FC of any fuzzy set from the family of $\tau - \mathfrak{K}$-FOSs (FCSs). Most of the properties of both approximations are similar, but some are different because of the nature of BFR as shown in Table 4.

$\tau - \mathfrak{K}$-FL and $\tau - \mathfrak{K}$-FU approximations are defined by $\tau - \mathfrak{K}$-FI and $\tau - \mathfrak{K}$-FC for any BFR $\mathfrak{K}$ from the collection of $\tau - \mathfrak{K}$-FOSs(FCSs), whereas in the fuzzy rough approximation, FI and the FU approximation are defined via fuzzy topology only when the BFR is reflexive and transitive.

4. Application for $\tau - \mathfrak{K}$-fuzzy open (closed) sets based decision making.

We purposed a new technique to decision making based on $\tau - \mathfrak{K}$-fuzzy open (closed) sets in this section. Based on soft fuzzy rough set, B. Sun and W. Ma [18] developed a decision-making method and also examined its validity. We are relying on the data in [18]. The results can be more objective, and the paradox for the same situation can be avoided. For illustration of results, consider the following case for the selection of house cited [18].

Let $\hat{U} = \{\mathfrak{j}_1, \mathfrak{j}_2, \mathfrak{j}_3, \mathfrak{j}_4, \mathfrak{j}_5, \mathfrak{j}_6\}$ be a set of six houses.

$\mathcal{P} = \{\mathfrak{p}_1$ (expensive), $\mathfrak{p}_2$ (beautiful), $\mathfrak{p}_3$ (wooden), $\mathfrak{p}_4$ (cheap), $\mathfrak{p}_5$ (green surroundings), $\mathfrak{p}_6$ (modern), $\mathfrak{p}_7$ (good repair) $\}$ be the parameters that any person is interested in buying a house as shown below in tabular form:

$\hat{U}/\mathcal{P}$	$\mathfrak{p}_1$	$\mathfrak{p}_2$	$\mathfrak{p}_3$	$\mathfrak{p}_4$	$\mathfrak{p}_5$	$\mathfrak{p}_6$	$\mathfrak{p}_7$
$\mathfrak{j}_1$	.3	.1	.4	.4	.1	.1	.5
$\mathfrak{j}_2$	.3	.3	.5	.1	.3	.1	.5
$\mathfrak{j}_3$	.4	.3	.5	.1	.3	.1	.6
$\mathfrak{j}_4$	.7	.4	.2	.1	.2	.1	.3
$\mathfrak{j}_5$	.2	.5	.2	.3	.5	.5	.4
$\mathfrak{j}_6$	.3	.5	.2	.2	.2	.3	.3

We'll present three different scenarios based on the parameters that a wealthy individual, a poor individual, or a middle-class person might use to determine which house is best suited to their needs.

Case 1. Rich person. Our approach to decision based on $\tau - \mathfrak{K}$-FOSs (FCSs) focus on the choice value β_i as follows: Assume set of houses as $\hat{U}_1 = \{\mathfrak{j}_2, \mathfrak{j}_3, \mathfrak{j}_4, \mathfrak{j}_5\}$ and parameters as $\mathcal{P}_1 = \{\mathfrak{p}_1, \mathfrak{p}_3, \mathfrak{p}_5, \mathfrak{p}_6\}$ that the rich person is interested in a house as shown in BFR $\mathfrak{K}$ as:

$$\mathfrak{K} = \begin{bmatrix} .3 & .5 & .3 & .1 \\ .4 & .5 & .3 & .1 \\ .7 & .2 & .2 & .1 \\ .2 & .2 & .5 & .5 \end{bmatrix}$$

$\mathfrak{l}_{\mathfrak{K}}(\mathfrak{p}_1) = \{\frac{.3}{\mathfrak{j}_2} + \frac{.4}{\mathfrak{j}_3} + \frac{.7}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}, \mathfrak{l}_{\mathfrak{K}}(\mathfrak{p}_3) = \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}, \mathfrak{l}_{\mathfrak{K}}(\mathfrak{p}_5) = \{\frac{.3}{\mathfrak{j}_2} + \frac{.3}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \mathfrak{l}_{\mathfrak{K}}(\mathfrak{p}_6) = \{\frac{.1}{\mathfrak{j}_2} + \frac{.1}{\mathfrak{j}_3} + \frac{.1}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}$ are called left $\mathfrak{K}$- relative fuzzy sets.

Let $\mathcal{Z}_1 = \{\frac{.5}{\mathfrak{j}_2}+\frac{.5}{\mathfrak{j}_3}+\frac{.7}{\mathfrak{j}_4}+\frac{.5}{\mathfrak{j}_5}\}$ be an optimum normal decision object on $\hat{U}_1$ and is the fuzzy set of houses with maximum membership values of attributes (parameters).

First, we calculate the fuzzy topology from left $\mathfrak{K}$- relative fuzzy sets as:

$\tau = \Big\{\{\frac{.3}{\mathfrak{j}_2} + \frac{.4}{\mathfrak{j}_3} + \frac{.7}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}, \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}, \{\frac{.3}{\mathfrak{j}_2} + \frac{.3}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.1}{\mathfrak{j}_2} + \frac{.1}{\mathfrak{j}_3} + \frac{.1}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}\{\{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.7}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}, \{\frac{.3}{\mathfrak{j}_2} + \frac{.4}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}, \{\frac{.3}{\mathfrak{j}_2} + \frac{.4}{\mathfrak{j}_3} + \frac{.7}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.3}{\mathfrak{j}_2} + \frac{.3}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}\{\{\frac{.1}{\mathfrak{j}_2} + \frac{.1}{\mathfrak{j}_3} + \frac{.1}{\mathfrak{j}_4} + \frac{.2}{\mathfrak{j}_5}\}, \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.7}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.3}{\mathfrak{j}_2} + \frac{.4}{\mathfrak{j}_3} + \frac{.2}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}\Big\}$.

Next, we calculate the complement of τ as follow:

$\tau^c = \Big\{\{\frac{.7}{\mathfrak{j}_2} + \frac{.6}{\mathfrak{j}_3} + \frac{.3}{\mathfrak{j}_4} + \frac{.8}{\mathfrak{j}_5}\}, \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.8}{\mathfrak{j}_4} + \frac{.8}{\mathfrak{j}_5}\}, \{\frac{.7}{\mathfrak{j}_2} + \frac{.7}{\mathfrak{j}_3} + \frac{.8}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.9}{\mathfrak{j}_2} + \frac{.9}{\mathfrak{j}_3} + \frac{.9}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.3}{\mathfrak{j}_4} + \frac{.8}{\mathfrak{j}_5}\}, \{\frac{.7}{\mathfrak{j}_2} + \frac{.6}{\mathfrak{j}_3} + \frac{.8}{\mathfrak{j}_4} + \frac{.8}{\mathfrak{j}_5}\}, \{\frac{.7}{\mathfrak{j}_2} + \frac{.6}{\mathfrak{j}_3} + \frac{.3}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.7}{\mathfrak{j}_2} + \frac{.7}{\mathfrak{j}_3} + \frac{.8}{\mathfrak{j}_4} + \frac{.8}{\mathfrak{j}_5}\}, \{\frac{.9}{\mathfrak{j}_2} + \frac{.9}{\mathfrak{j}_3} + \frac{.9}{\mathfrak{j}_4} + \frac{.8}{\mathfrak{j}_5}\}, \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.8}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.5}{\mathfrak{j}_2} + \frac{.5}{\mathfrak{j}_3} + \frac{.3}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}, \{\frac{.7}{\mathfrak{j}_2} + \frac{.6}{\mathfrak{j}_3} + \frac{.8}{\mathfrak{j}_4} + \frac{.5}{\mathfrak{j}_5}\}\Big\}$.

Now, we calculate the $\tau - \mathfrak{K}$-fuzzy lower and $\tau - \mathfrak{K}$-fuzzy upper approximations from the collection of $\tau - \mathfrak{K}$-fuzzy open(closed) sets and also calculate the choice value $\beta_i = \frac{\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)+\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)}{2}$, which can be used for right decision.

Finally, the optimum decision for the problem is the house $\mathfrak{j}_i$ having choice value β_i maximum. We choose any object $\mathfrak{j}_i \in \hat{U}_1$ to be optimum decision for the case with same maximum value β_i. Clearly from table 5, β_3 is maximum choice value, scored by $\mathfrak{j}_4$. Therefore, the rich person will select the house $\mathfrak{j}_4$.

Case 2. Similarly, for financially weak Person, we consider, $\hat{U}_2 = \{\mathfrak{j}_1, \mathfrak{j}_5\}$ is the set of houses and $\mathcal{P}_2 = \{\mathfrak{p}_1, \mathfrak{p}_4, \mathfrak{p}_7\}$ as the set of parameters and calculate the choice value β_i for the collection of subsets of $\mathcal{Z}_2 = \{\frac{.5}{\mathfrak{j}_1} + \frac{.4}{\mathfrak{j}_5}\}$. The decision is for the house $\mathfrak{j}_1$ based on the maximum choice value.

Case 3. Let $\hat{U}_3 = \{\mathfrak{j}_1, \mathfrak{j}_5, \mathfrak{j}_6\}$ is the set of houses and $\mathcal{P}_3 = \{\mathfrak{p}_1, \mathfrak{p}_4, \mathfrak{p}_5, \mathfrak{p}_7\}$ as the set of parameters for middle-class person and calculate the choice value β_i for the collection of subsets of $\mathcal{Z}_3 = \{\frac{.5}{\mathfrak{j}_1} + \frac{.5}{\mathfrak{j}_5} + \frac{.3}{\mathfrak{j}_6}\}$. Based on β_i, the choice is for home $\mathfrak{j}_1$ or $\mathfrak{j}_5$. The person can choose either of the houses from $\mathfrak{j}_1$ or $\mathfrak{j}_5$ due to the same choice value.

Algorithm for $\tau-\mathfrak{K}$-fuzzy approximation based on $\tau-\mathfrak{K}$-fuzzy open (closed) Sets:
Input: $\hat{U}, \mathcal{P}, \mathcal{Z}_1$ and $\mathfrak{K}$, where $\mathfrak{K}$ is binary fuzzy relation as defined in section 2(definition 2.2).
Output: Best choice of $\mathfrak{j}_i \in \hat{U}$.
Step 1. Using definition 2.7, construct the left $\mathfrak{K}$- relative fuzzy sets.
Step 2. Construct the fuzzy topology τ from $\mathfrak{l}_{\mathfrak{K}}$ such that $\mathfrak{l}_{\mathfrak{K}}$ satisfy all the axioms of fuzzy topology (definition 2.3).
Step 3. Compute the complement of τ.
Step 4. Compute $\tau-\mathfrak{K}$-fuzzy open(closed) sets for the collection of subsets of $\mathcal{Z}_1$ denoted as $\mathcal{A}$.
Step 5. Compute $\tau-\mathfrak{K}$-fuzzy lower and $\tau-\mathfrak{K}$-fuzzy upper approximations for $\mathcal{A}_i \subseteq \mathcal{A}$ using the definition 3.12.
Step 6. Compute choice value $\beta_i = \frac{\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)+\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)}{2}$.
Step 7. Compute $\beta' = max(\beta_i)$, where i is the number of houses.
Step 8. Decision is $\mathfrak{j}_k \in \hat{U}$, if $\beta_k = \beta'$.
Step 9. Choose any one of $\mathfrak{j}_k$ for more than one value of k.

TABLE 4.

Sr. No.	**$\tau-\mathfrak{K}$-fuzzy approximation** Let $(\hat{U}, \tau, \mathfrak{K})$ be FTAS and $\mathfrak{K}$ be BFR on $\hat{U}$ and $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are two fuzzy subsets of $\hat{U}$.	**Fuzzy rough approximation**[4, 14] Let $\hat{U}$ be a universal set, $\mathfrak{K}$ be BFR on $\hat{U}$ and $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are two fuzzy subsets of $\hat{U}$.
i	$\mathcal{Z}_1 \subseteq \mathcal{Z}_2 \Rightarrow \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$ and $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$	$\mathcal{Z}_1 \subseteq \mathcal{Z}_2 \Rightarrow \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$ and $\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$
ii	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = \left(\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1^c)\right)^c$	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = \left(\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1^c)\right)^c$
iii	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = \left(\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1^c)\right)^c$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) = \left(\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1^c)\right)^c$
iv	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$
v	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2) = \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2) = \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$
vi	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cup \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2) \subseteq \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$
vii	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \cap \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_2)$
viii	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\phi) = \phi = \overline{\mathfrak{K}}_{\mathfrak{K}}(\phi)$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\phi) = \phi$
ix	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\hat{U}) = \hat{U} = \overline{\mathfrak{K}}_{\mathfrak{K}}(\hat{U})$	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\hat{U}) = \hat{U}$
x	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \mathcal{Z}_1 \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1) \subseteq \mathcal{Z}_1 \subseteq \overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$, if $\mathfrak{K}$ is reflexive.
xi	$\underline{\mathfrak{K}}_{\mathfrak{K}}\left(\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)\right) = \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$	$\underline{\mathfrak{K}}_{\mathfrak{K}}\left(\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)\right) = \underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{Z}_1)$, if $\mathfrak{K}$ is reflexive and transitive.

Here, "–" denotes no need of values.

TABLE 5.

$\mathcal{A} = \mathcal{P}(\mathcal{Z}_1)$	$\tau - \mathfrak{K}$-FOS or not	$\tau - \mathfrak{K}$-FCS	$\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)$	$\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)$	$\beta_i = \frac{\underline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)+\overline{\mathfrak{K}}_{\mathfrak{K}}(\mathcal{A}_i)}{2}$
$\left\{\frac{.5}{j_2}+\frac{0}{j_3}+\frac{0}{j_4}+\frac{0}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{1}{j_3}+\frac{1}{j_4}+\frac{1}{j_5}\right\}$	$\left\{\frac{.5}{j_2}+\frac{0}{j_3}+\frac{0}{j_4}+\frac{0}{j_5}\right\}$	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{.3}{j_4}+\frac{.5}{j_5}\right\}$	$\beta_1 = \left\{\frac{.5}{j_2}+\frac{.25}{j_3}+\frac{.15}{j_4}+\frac{.25}{j_5}\right\}$
$\left\{\frac{0}{j_2}+\frac{.5}{j_3}+\frac{0}{j_4}+\frac{0}{j_5}\right\}$	Yes	$\left\{\frac{1}{j_2}+\frac{.5}{j_3}+\frac{1}{j_4}+\frac{1}{j_5}\right\}$	$\left\{\frac{0}{j_2}+\frac{.5}{j_3}+\frac{0}{j_4}+\frac{0}{j_5}\right\}$	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{.3}{j_4}+\frac{.5}{j_5}\right\}$	$\beta_2 = \left\{\frac{.25}{j_2}+\frac{.5}{j_3}+\frac{.15}{j_4}+\frac{.25}{j_5}\right\}$
$\left\{\frac{0}{j_2}+\frac{0}{j_3}+\frac{.7}{j_4}+\frac{0}{j_5}\right\}$	Yes	$\left\{\frac{1}{j_2}+\frac{1}{j_3}+\frac{.3}{j_4}+\frac{1}{j_5}\right\}$	$\left\{\frac{0}{j_2}+\frac{0}{j_3}+\frac{.7}{j_4}+\frac{0}{j_5}\right\}$	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{1}{j_4}+\frac{.5}{j_5}\right\}$	$\beta_3 = \left\{\frac{.25}{j_2}+\frac{.25}{j_3}+\frac{.85}{j_4}+\frac{.25}{j_5}\right\}$
$\left\{\frac{0}{j_2}+\frac{0}{j_3}+\frac{0}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{1}{j_2}+\frac{1}{j_3}+\frac{1}{j_4}+\frac{.5}{j_5}\right\}$	$\left\{\frac{0}{j_2}+\frac{0}{j_3}+\frac{0}{j_4}+\frac{.5}{j_5}\right\}$	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{.3}{j_4}+\frac{.5}{j_5}\right\}$	$\beta_4 = \left\{\frac{.25}{j_2}+\frac{.25}{j_3}+\frac{.15}{j_4}+\frac{.5}{j_5}\right\}$
$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{0}{j_4}+\frac{0}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{1}{j_4}+\frac{1}{j_5}\right\}$	-	-	-
$\left\{\frac{.5}{j_2}+\frac{0}{j_3}+\frac{.7}{j_4}+\frac{0}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{1}{j_3}+\frac{.3}{j_4}+\frac{1}{j_5}\right\}$	-	-	-
$\left\{\frac{.5}{j_2}+\frac{0}{j_3}+\frac{0}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{1}{j_3}+\frac{1}{j_4}+\frac{.5}{j_5}\right\}$	-	-	-
$\left\{\frac{0}{j_2}+\frac{.5}{j_3}+\frac{.7}{j_4}+\frac{0}{j_5}\right\}$	Yes	$\left\{\frac{1}{j_2}+\frac{.5}{j_3}+\frac{.3}{j_4}+\frac{1}{j_5}\right\}$	-	-	-
$\left\{\frac{0}{j_2}+\frac{.5}{j_3}+\frac{0}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{1}{j_2}+\frac{.5}{j_3}+\frac{1}{j_4}+\frac{.5}{j_5}\right\}$	-	-	-
$\left\{\frac{0}{j_2}+\frac{0}{j_3}+\frac{.7}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{1}{j_2}+\frac{1}{j_3}+\frac{.3}{j_4}+\frac{.5}{j_5}\right\}$	-	-	-
$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{.7}{j_4}+\frac{0}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{.3}{j_4}+\frac{1}{j_5}\right\}$	-	-	-
$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{0}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{1}{j_4}+\frac{.5}{j_5}\right\}$	-	-	-
$\left\{\frac{.5}{j_2}+\frac{0}{j_3}+\frac{.7}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{1}{j_3}+\frac{.3}{j_4}+\frac{.5}{j_5}\right\}$	-	-	-
$\left\{\frac{0}{j_2}+\frac{.5}{j_3}+\frac{.7}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{1}{j_2}+\frac{.5}{j_3}+\frac{.3}{j_4}+\frac{.5}{j_5}\right\}$	-	-	-
$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{.7}{j_4}+\frac{.5}{j_5}\right\}$	Yes	$\left\{\frac{.5}{j_2}+\frac{.5}{j_3}+\frac{.3}{j_4}+\frac{.5}{j_5}\right\}$	-	-	-

5. **Conclusion.** The concept of fuzzy topology formed by binary fuzzy relations is quite useful in fuzzy domain. In this paper, new fuzzy approximations i.e. $\tau - \mathfrak{K}$-fuzzy approximations are introduced. $\tau - \mathfrak{K}$-FOSs (FCSs) characterizes these fuzzy approximations. We started by creating a collection of all $\tau - \mathfrak{K}$-FOSs (FCSs) of any fuzzy subset of the universal set $\hat{U}$. The new fuzzy topology may or may not be formed from this collection. $\tau - \mathfrak{K}$-FL and $\tau - \mathfrak{K}$-FU approximations are defined from

a collection of $\tau - \mathfrak{K}$-FOSs (FCSs) and these approximations are also defined for α-level sets. We compared the properties of $\tau - \mathfrak{K}$-fuzzy approximations based on $\tau - \mathfrak{K}$ -FOSs (FCSs) with fuzzy rough approximations. In Table 4, we have enlisted eleven properties, where properties (i) to (vii) are true for any BFR whereas properties (viii), (ix), (x), (xi) reveal the difference between both the approximations. Also we have described a practical application based on new fuzzy approximations. We can investigate the properties of new fuzzy approximations for similarity BFRs in the future and can modify the proposed decision method.

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